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On the Metric Properties of the Feigenbaum Attractor

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The two standard literature definitions of the function associated with the Feigenbaum attractor are not equivalent. The method due to Vul *et al.* and Feigenbaum is used to calculate the Haussdorff dimension of the Feigenbaum attractor, using as input the trajectory scaling functions. The two calculations yield the same Hausdorff dimension $D = 0.5380451435$ to within the accuracy of the computation.

KEY WORDS: Period-doubling; fractal dimensions; trajectory scaling functions.

1. INTRODUCTION

The Feigenbaum attractor is a fractal object for which the Hausdorff dimension and generalized fractal dimensions have been computed by a variety of methods.[1-8] In this article I will describe in some detail a new method due to Feigenbaum[4] to carry through these calculations based on the thermodynamic analogy of Vul *et al.*[10] The preliminary results of the calculation have been reported elsewhere.[8]

I will also show that the different constructions given in the literature are not equivalent, even though they lead to identical values for the fractal dimensions. For the sake of illustration a set of new constructions is introduced that provide an interpolation between the two.

2. THE FEIGENBAUM CONSTRUCTION[3]

A pictorial illustration of this construction is given in Fig. 1.

Here I briefly restate the original construction of the period-doubling attractor given by Feigenbaum.[3] Consider a family of unimodal maps on

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the interval, $f_p(x)$, that goes through a series of period-doublings as the parameter p is varied. These maps have one critical point, and if a periodic orbit includes this point, it is said to be superstable. We let the parameter p_i be the value of p for which the system has a superstable 2^i-cycle; i.e., p_0 indicates a superstable fixpoint, p_1 indicates a superstable 2-cycle, p_2 a

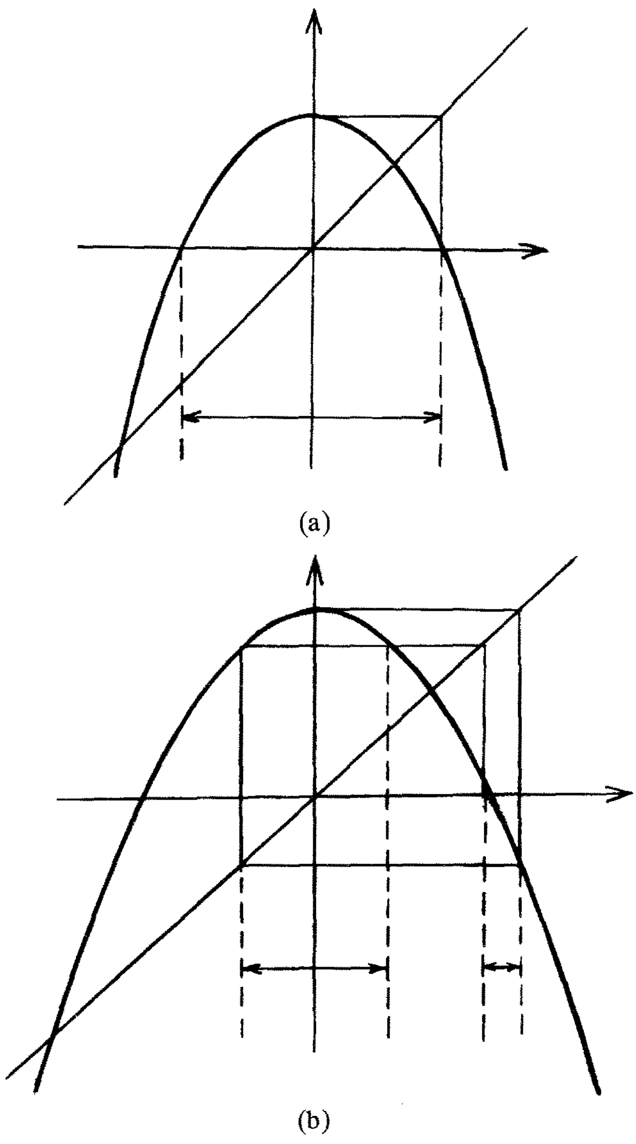

Fig. 1. The Feigenbaum construction. (a) Level 0, one interval. (b) Level 1, two intervals.

superstable 4-cycle, etc. These values accumulate at a value p_∞, with asymptotic convergence as δ^{-n}, $\delta = 4.6690216\ldots$. At p_∞ we have a "2^∞-cycle," and by the Feigenbaum attractor we mean the set of points generated by successive iterations of the critical point at this parameter value.

Consider now the following construction of a Cantor-like set: at level m the interval $\Delta_0^{(m)}$ is the interval

$$\Delta_0^{(m)} = [-|f_{pm+1}^{2^m}(0)|, +|f_{pm+1}^{2^m}(0)|] \tag{2.1}$$

The jth interval $\Delta_j^{(m)}$ is then $f_{pm+1}^{j}(\Delta_0^{(m)})$. We have

$$\Delta_0^{(m)} \supset f_{pm+1}^{2^m}(\Delta_0^{(m)}), \qquad \Delta_j^{(m)} = f_{pm+1}^{2^m}(\Delta_j^{(m)})$$

To every interval $\Delta_j^{(m)}$ at level m we associate two intervals on level $m+1$: $\Delta_j^{(m+1)}$ and $\Delta_{j+2^m}^{(m+1)}$. Notice that $\Delta_j^{(m+1)}$ and $\Delta_{j+2^m}^{(m+1)}$ need not be subsets of $\Delta_j^{(m)}$, as is obvious if $f_p(x) = p - x^2$ and one considers $\Delta_1^{(m+1)}$ versus $\Delta_1^{(m)}$. To every interval $\Delta_j^{(m)}$ there is thus associated a unique sequence of intervals, such that every interval in the sequence has a unique predecessor. If $j = i_1 2^{m-1} + i_2 2^{m-2} + \cdots + i_m$, $i_r = 0, 1$, these intervals are, in level one, the interval number i_m, in level two, the interval number $i_{m-1}2 + i_m$, in level three, the interval number $i_{m-2}4 + i_{m-1}2 + i_m$, etc. As $m \to \infty$, the number of intervals grows as 2^m and converges to the orbit of $f_{p\infty}(0)$ in the following sense: let $\{x(j)\}_0^m$ be a sequence of numbers such that $x(j)_k$ belongs to the kth interval in the chain associated with $\Delta_j^{(m)}$. Then

$$\begin{aligned} &|f_{p\infty}^{j}(0) - x(j)_m| \\ &\quad \leqslant \max\{|f_{p\infty}^{j}(0) - f_{pm+1}^{j}(0)|, |f_{p\infty}^{j}(0) - f_{pm+1}^{j+2^m}(0)|\} \\ &\quad \leqslant |f_{p\infty}^{2^{m+1}}(0)| \sim |1/\alpha|^{m+1} \end{aligned} \tag{2.2}$$

where $\alpha = -2.5029\ldots$ is the Feigenbaum constant.

Now introduce the directed length $d_j^{(m)}$ of the interval $\Delta_j^{(m)}$ defined as

$$d_j^{(m)} \equiv f_{pm+1}^{j}(0) - f_{pm+1}^{j+2^m}(0) \tag{2.3}$$

and extended to all j with the convention $d_{j+2^m}^{(m)} = -d_j^{(m)}$.

The quotient

$$\sigma^{(m-1)}(j/2^{m+1}) \equiv d_j^{(m)}/d_j^{(m-1)} \tag{2.4}$$

is the Feigenbaum trajectory scaling function, which we proceed to study.

Let $d_j^{(m-1)}$ be an interval far from zero in the sense that

$$|d_j^{(m-1)} \ll |f_{pm}^{j}(0)|$$

Then

$$d_{j+1}^{(m-1)} = f_{pm}(f_{pm}^{j}(0)) - f_{pm}(f_{pm}^{j+2^{m-1}}(0))$$
$$= f_{pm}(f_{pm}^{j}(0)) - f_{pm}(f_{pm}^{j}(0) - d_j^{(m-1)})$$
$$\approx f'_{pm}(f_{pm}^{j}(0))\ d_j^{(m-1)} \tag{2.5}$$

Hence

$$\sigma^{(m-1)}((j+1)/2^{m+1}) = d_{j+1}^{(m)}/d_{j+1}^{(m-1)}$$
$$\approx \{f'_{pm+1}(f_{pm+1}^{j}(0))/f'_{pm}(f_{pm}^{j}(0))\}\,\sigma^{(m-1)}(j_{2^{m+1}}) \tag{2.6}$$

In the limit $m \to \infty$ the factor in front goes to one, so in that limit, scaling only change upon close passage to the origin. Consider therefore the scaling functions $\sigma^{(m+k-1)}(j \cdot 2^k/2^{m+k+1})$, and k very large. We can now make use of Feigenbaum universality[3,9]:

$$\alpha^k f_{pm+k}^{2^k}(x/\alpha^k) \to \mu_f\, g_m(x/\mu_f) \qquad \text{as} \quad k \to \infty \tag{2.7}$$

$\alpha = -2.5209...$ is the period-doubling constant, g_m is a universal function, and μ_f is a scale depending on the function family $f_p(x)$. One sees from the definitions that $g_m(x)$ owns a superstable 2^m-cycle and that the functions g_m satisfy a recursion relation; $\alpha g_m \circ g_m(x/\alpha) = g_{m-1}(x)$. We also have

$$f_{p\infty}^{2^{m+1}}(0) = f_{p\infty+(m+1)}^{2^{m+1}}(0) \to (1/\alpha^{m+1})\,\mu_f g(0)$$

where g satisfies the Cvitanović–Feigenbaum functional equation: $\alpha g^2(x/\alpha) = g(x)$. We have

$$\sigma^{(m+k-1)}(j2^k/2^{m+k+1}) = \frac{\alpha^k f_{pm+k+1}^{j2^k}(0/\alpha^k) - \alpha^k f_{pm+k+1}^{j2^k+2^{k+m}}(0/\alpha^k)}{\alpha^k f_{pm+k}^{j2^k}(0/\alpha^k) - \alpha^k f_{pm+k}^{j2^k+2^{k+m-1}}(0/\alpha^k)}$$
$$\underset{k\to\infty}{=} \frac{\mu_f g_{m+1}^{j}(0) - \mu_f g_{m+1}^{j+2^m}(0)}{\mu_f g_m^{j}(0) - \mu_f g_m^{j+2^{m-1}}(0)}$$
$$= \frac{g_{m+1}^{j}(0) - g_{m+1}^{j}((1/\alpha^m)\, g_1(0))}{g_m^{j}(0) - g_m^{j}((1/\alpha^{m-1})\, g_1(0))} \tag{2.8}$$

The normalization of the function family is conveniently chosen such that $g_1(0) = 1$, and so

$$\sigma(j/2^{m+1}) = \lim_{k\to\infty} \sigma^{(m+k+1)}(j2^k/2^{m+k+1})$$
$$= \frac{g_{m+1}^{j}(0) - g_{m+1}^{j}(1/\alpha^m)}{g_m^{j}(0) - g_m^{j}(1/\alpha^{m-1})} \tag{2.9}$$

The function $\sigma(j/2^{m+1})$ is a universal function and gives the 2^{m+1} values at which the original scaling function $\sigma^{(m+k-1)}(i/2^{m+k+1})$ changes the most as $k \to \infty$. One can therefore regard it as the values at 2^{m+1} points of the full asymptotic scaling function $\sigma(t)$. If we extend the function $\sigma(j/2^{m+1})$ to the line between the points $j/2^{m+1}$, as a piecewise constant function of t, we have the approximation to $\sigma(t)$ in level m.

Now express j as $j = i_0 2^m + i_1 2^{m-1} + \cdots + i_m \equiv (i_0, i_1,..., i_m)$; $i_r = 0, 1$. The complement of 1 is 0 and vice versa. Then

$$\begin{aligned} g_m^j(0) &= g_m^{i_m} \circ g_m^{i_{m-1}2^1} \circ g_m^{i_{m-2}2^2} \circ \cdots \circ g_m^{i_0 2^m}(0) \\ &= g_m^{i_m}(1/\alpha) \circ \alpha g_m^{i_{m-1}2^1}(1/\alpha) \circ (1/\alpha) \circ (\alpha^2)\, g_m^{i_{m-2}2^2}(1/\alpha^2) \\ &\quad \circ (1/\alpha) \circ \cdots \circ (\alpha^m)\, g_m^{j_0 2^m}(0) \\ &= g_m^{i_m}(1/\alpha) \circ g_{m-1}^{i_{m-1}}(1/\alpha) \circ g_{m-2}^{i_{m-2}}(1/\alpha) \circ \cdots \circ g_0^{i_0}(0) \end{aligned} \tag{2.10}$$

If we introduce the notation

$$A(i_1,..., i_m) = g_m^{i_m}(1/\alpha) \circ g_{m-1}^{j_{m-1}}(1/\alpha) \circ \cdots \circ g_1^{i_1}(0) \tag{2.11}$$

we can write in a compact form

$$\sigma(i_0, i_1,..., i_m) = \frac{A(i_0, i_1,..., i_m) - A(i_0', i_1,..., i_m)}{A(i_1, i_2,..., i_m) - A(i_1', i_2,..., i_m)} \tag{2.12}$$

where i_0' is the complement of i_0 and i_1' is the complement of i_1. The notation shows that to calculate any component of the mth level approximation to $\sigma(t)$ we need at most $m+1$ functional compositions, and not, as it would seem, 2^m. Furthermore, each of these compositions acts on an argument close to zero where the absolute value of the derivative is less than one, and so errors do not build up. The error in A is thus roughly proportional to the error in the determination of the function applied last, $g_m(x)$ (provided $i_m = 1$).

The functions $\sigma(j/2^{m+1})$ converge uniformly to $\sigma(t)$. The function $\sigma(t)$ suffers a discontinuity on every dyadic rational and we denote by $\sigma(i_0,..., i_m, +)$ the value of σ immediately after the discontinuity at $j = (i_0,..., i_m)$. It is straightforward to derive the following statements. (Derivations are given in Appendix A.)

A. $2^{-(m+1)}\{\sigma(i_0,..., i_m, +) - \sigma(i_0,..., i_m)\} \to 0$ as $m \to \infty$ if $i_m = 1$. That is, the jumps go to zero exponentially faster than the distance between the jumps.

B. $2^{-(m+1)}\{\sigma(i_0,..., u_k,..., i_m, 1\} - \sigma(i_0,..., i_k,..., i_m, 0)\} \to 0$ as $m \to \infty$ if $\alpha^k \gg 2^m$, $k \gg m(\log 2/\log \alpha)$, where i_k is the last index in the sequence that is different from zero. The numbers that do not have this kind of binary expansion have measure zero and so the function σ is both right continuous and right differentiable almost everywhere with $(d/dt^+)\,\sigma(t)=0$ almost everywhere.

C. $2^{-(m+2)}\{\sigma((2j-1)/2^{m+2})-\sigma(2j/2^{m+2})\}\to 0$ as $m\to\infty$ regardless of j. Hence, the function $\sigma(t)$ is everywhere left continuous and left differentiable with $(d/dt^-)\,\sigma(t)=0$.

D. If we approximate $\sigma(t)$ at level m on the interval $[j/2^{m+1}, j+1/2^{m+1}]$ either as $\sigma(j/2^{m+1}+)$ or as $\sigma((j+1)/2^{m+1})$, the average change as we introduce the next level, $m+1$, behaves as a sequence with two transients, roughly as

$$\begin{aligned} &2^{-(m+1)}\left\{\sum_{(i_0,\ldots,i_m)} \alpha^{-(m+1)}(\alpha^{-1})^{n(i_0,\ldots,i_m)}+\delta^{-m}\right\} \\ &\quad = 2^{-(m+1)}(\alpha^{-1}+\alpha^{-2})^m+\delta^{-m} \\ &\quad \approx (-8)^{-m}+\delta^{-m} \end{aligned}$$

$n(i_0,\ldots, i_m)$ is the number of zeros in the sequence $(i_0,\ldots, i_m)$.

3. THE CONSTRUCTION OF VUL, SINAI, AND KHANIN[10]

A pictorial illustration of this construction is given in Fig. 2.

We will consider the following construction of a Cantor-like set: at level m the interval $\tilde{\Delta}_0^{(m)}$ is the interval

$$\tilde{\Delta}_0^{(m)}=[-|f_{p\infty}^{2^m}(0)|,\ |f_{p\infty}^{2^m}(0)|] \tag{3.1}$$

The interval $\tilde{\Delta}^{(m)}{}_j$ is then $f_{p\infty}^j(\tilde{\Delta}_0^{(m)})$. We have that

$$\tilde{\Delta}_0^{(m)}\supset f_{p\infty}^{2^m}(\tilde{\Delta}_0^{(m)}) \tag{3.2}$$

To every interval $\tilde{\Delta}_j^{(m)}$ there are now associated two subintervals; $\tilde{\Delta}_j^{(m+1)}$ and $\tilde{\Delta}_{j+2^m}^{(m+1)}$. These intervaals clearly converge to the orbit of 0 at $p=p_\infty$. We can introduce directed lengths of these intervals and scaling functions as quotients between lengths of intervals and arrive at a universal scaling function

$$\sigma^{\mathrm{VSK}}(j/2^{m+1})=\frac{g^j(0)-g^{j+2^m}(0)}{g^j(0)-g^{j+2^{m-1}}(0)} \tag{3.3}$$

Here g is the universal function satisfying the Cvitanovic'–Feigenbaum functional equation. The standard normalization is $g(0)=1$, as in the high-accuracy approximation of g given by Lanford.[11]

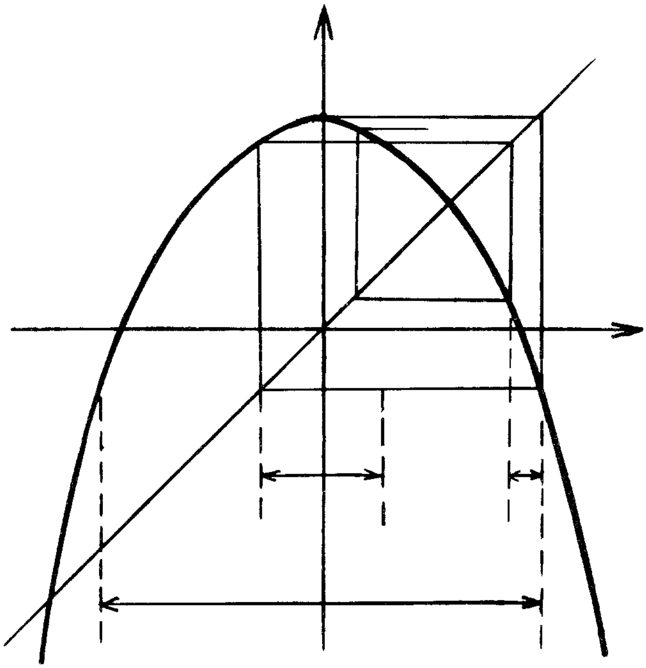

Fig. 2. The construction of Vul *et al.*

If $j = i_0 2^m + i_1 2^{m-1} + \cdots + i_m$ and we use the notation

$$B(i_0,\dots, i_m) = g^{i_m}(1/\alpha) \circ g^{i_{m-1}}(1/\alpha) \circ \cdots \circ g^{i_0}(0) \tag{3.4}$$

we may write in a compact form

$$\sigma^{\text{VSK}}(i_0,\dots, i_m) = \frac{B(i_0,\dots, i_m) - B(i'_0,\dots, i_m)}{B(i_1,\dots, i_m) - B(i'_1,\dots, i_m)} \tag{3.5}$$

where i'_0 is the complement of i_0 and i'_1 is the complement of i_1.

All the statements concerning the functions A and σ in the last section carry over on the new functions B and σ^{VSK}, the only difference being that this time there are no factors $O(\delta^{-m})$. The construction with σ converges down the unstable manifold to the fixpoint of the period-doubling operator, while the construction with σ^{VSK} converges transversely along the stable manifold. Therefore, we have the full equivalence of the statements A, B, C, and D, and for D we can make the stronger statement that the average change in $\sigma^{\text{VSK}}(t)$, as we introduce level $m+1$, behaves as $[(1/2\alpha)(1+1/\alpha)]^m \approx (-8)^{-m}$, no corrections coming from $O(\delta^{-m})$ terms.

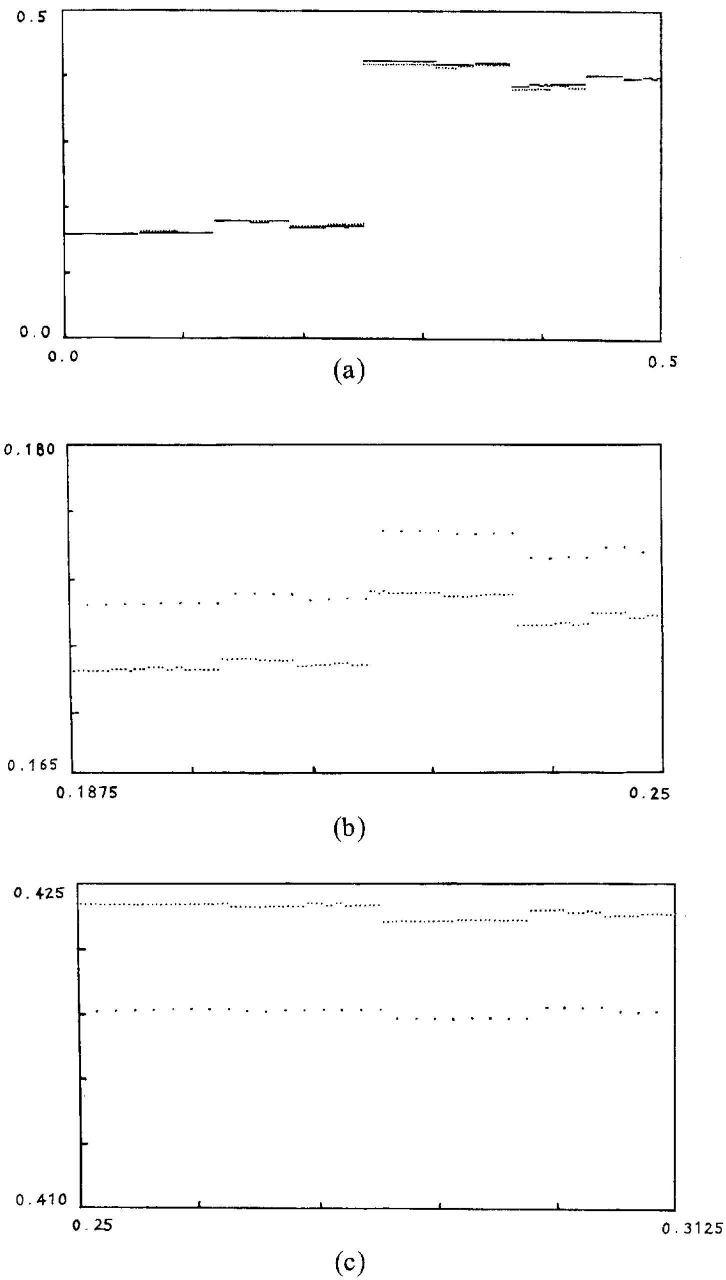

Fig. 3. (a) The two scaling functions on the interval 0–0.5. The construction of Vul *et al.* is the more densely dotted. (b) The two scaling functions on the interval 0.1875–0.25. (c) The two scaling functions on the interval 0.25–0.3125.

4. THE NUMERICAL EVALUATION OF THE SCALING FUNCTIONS

As the scaling functions σ and σ^{VSK} are given by analytically different expressions, there is no *a priori* reason to suspect that they are identical. To compare, one need to compute numerically the functions $g_1, g_2, ..., g_\infty = g$ with the same normalization. This is done by solving the functional equation of Vul *et al.* for the function family $g_p(x)$ and the constants α and δ,[10]

$$g_p(x) = \alpha g_{1+p/\delta} \circ g_{1+p/\delta}(x/\alpha) \tag{4.1}$$

with normalizations

$g_0(0) = 0$ (superstable fixpoint)

$g_1^2(0) = 0$ (superstable 2-cycle)
(these fix the origin and set the scale in the variable p)

$g_1(0) = 1$ (sets the scale in the variable x)

The function $g_i(x)$ is defined as $g_p(x)$ at the parameter value for which the function has a superstable 2^p-cycle; $p_i = \{(1-(1/\delta)^i)/(1-(1/\delta))\}$. This functional equation has a stable fixpoint which is the unstable manifold to the functional equation of Cvitanovic' and Feigenbaum. Numerically this can be solved by iteration of the transformation and truncation.

I solved it using a 40×40 matrix representation of the expansion $g_p(x) = \sum g_{ij} p^i x^{2j}$ to accuracy better than one part in 10^{-11}. This is then about the accuracy to which I can calculate σ on rationals with short expansions in base 2.

One then finds (Fig. 3) that though the two scaling functions as expected agree close to $t = 0$ and $t = 1/2$, they differ around $t = 1/4$ [$\sigma(1/4) = 0.1752...$, but $\sigma^{VSK}(1/4) = 0.1722...$, and $\sigma(1/4+) = 0.4190...$, but $\sigma^{VSK}(1/4+) = 0.4240...$], and around every other dyadic rational.

5. FURTHER CONSTRUCTIONS

One can construct family of scaling functions in the following way: On level m let the interval $\Delta_0^{(m)}$ be the interval

$$\Delta_0^{(m)} = [-|f_{pm+N}^{2^m}(0)|; |f_{pm+N}^{2^m}(0)|] \tag{5.1}$$

and $\Delta_j^{(m)} = f_{pm+N}^j(\Delta_0^{(m)})$. Now everything follows as before and one finds a universal trajectory scaling function

$$\sigma^N(j/2^{m+1}) = \frac{g_{m+N}^j(0) - g_{m+N}^j(\alpha^{-m} g_N(0))}{g_{m+N-1}^j(0) - g_{m+N-1}(\alpha^{-m+1} g_N(0))} \tag{5.2}$$

Here we choose the normalization $g_N(0)=1$ and introduce for $j=(i_0,..., i_m)$

$$A^N(i_0,..., i_m)=g_{N+m}^{i_m}\alpha^{-1}\circ\cdots\circ g_{N+1}^{i_1}\alpha^{-1}\circ g_N^{i_0}(0) \tag{5.3}$$

We can then write

$$\sigma^N(i_0,..., i_m)=\frac{A^N(i_0,..., i_m)-A^N(i'_0,..., i_m)}{A^N(i_1,..., i_m)-A^N(i'_1,..., i_m)} \tag{5.4}$$

Obviously $A^1=A$, $\sigma^1=\sigma$ (the Feigenbaum construction), and $A^\infty=B$, $\sigma^\infty=\sigma^{\mathrm{VSK}}$ (the Vul, Sinai, and Khanin construction), so these scaling functions provide an interpolation between the usual ones.

6. FRACTAL DIMENSIONS AND SCALING INDICES

In this section I calculate the generalized fractal dimensions of the Feigenbaum attractor from the trajectory scaling functions. The formula for the fractal dimension[5,6] with index q, D_q, as the limit as $m\to\infty$ of a sum over intervals i on level m with lengths l_i and probabalities p_i

$$\sum_{i=0}^{2^m-1}\frac{(p_i)^q}{(l_i)^{\tau_q}}=1; \qquad \tau_q=(q-1)D_q \tag{6.1}$$

can be rewritten following Vul *et al.*[10] and Feigenbaum[4] as a partition function over a 1D spin system. Namely choose, as is natural, the intervals to be $\Delta_i^{(m)}$ which all have weight 2^{-m}, $F(\beta)=-q$, $\beta=-\tau_q$, and $W(i)=-\ln|d_i^{(m)}|$.

Then

$$2^{-mF_m(\beta)}=\sum_{(i_0,...,i_m)} e^{-\beta W(i)}; \qquad i=(i_0,..., i_m) \tag{6.2}$$

$$F(\beta)=\lim_{N\to\infty} F_N(\beta) \tag{6.3}$$

Now

$$|d_1^{(m)}|\approx|\sigma(i_0,..., i_m)|\,|d_1^{(m-1)}|\approx|\sigma(i_0,..., i_m)|\cdots|\sigma(i_m, 0,..., 0)|\,|d_0^{(0)}| \tag{6.4}$$

The interaction potential $W(i)$ of the spin configuration $i=(i_0,..., i_m)$ can be divided into interactions between neighboring spins, next-neighbor spins, and so on. Due to property D of σ and σ^{VSK} this interaction decreases exponentially with distance, so one knows from general theorems in statistical mechanics[10,13] that in the limit as $m\to\infty$, $F(\beta)$ is a smooth, monotonically increasing function of β.

The 2^{m+1} elements in the mth-level approximation to σ thus build up a $2^{m-1} \times 2^{m-1}$ transfer matrix (the first bit in the address only contributes a sign, so it does not enter):

$$T_m(\beta) = \begin{pmatrix} |\sigma(0,0,0,...,0)|^\beta, & |\sigma(0,0,0,...,1)|^\beta, & 0, 0,... & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ |\sigma(0,1,0,...,0)|^\beta, & |\sigma(0,1,0,...,1)|^\beta, & 0,0,... & \\ 0 & 0 & & \\ \vdots & \vdots & ..., 0, 0, |\sigma(0,1,1,...,0)|^\beta, & |\sigma(0,1,1,...,1)|^\beta \end{pmatrix} \tag{6.5}$$

As $m \to \infty$, we have $F(\beta) = -\ln \lambda(\beta)/\ln 2$, where $\lambda(\beta)$ is the largest eigenvalue of $T_m(\beta)$. As pointed out by Feigenbaum,[4] it is obvious that just a part of the structure of σ enters the quantity $F(\beta)$. Suppose one is on level 2 and reads the binary address of an interval

$$i = (i_0,..., 0, 1, 0, 1, 0, 0, 0,..., i_m)$$

Then the substrings 01 and 10 occur on opposite sides of a block of 1's, so asymptotically they occur with equal frequency. Therefore the scaling functions $\sigma(0, 1)$ and $\sigma(1, 0)$ only enter through the combination $\sigma(0, 1)\,\sigma(1, 0)$. Similarly, on level 3 there are only five independent quantities entering $F(\beta)$, as the substrings 001 and 100, as well as 011 and 110, occur with equal frequency. Furthermore, there is a relation $f(101) - f(010) = f(011) - f(001)$ between the frequencies of occurences of substrings, which brings the number of independent components down to five.

There is thus an explicit way to calculate D_q given σ, and from D_q one can calculate their Legendre transformations, the scaling indices of Halsey *et al.*[6]

7. NUMERICAL RESULTS AND CONVERGENCE

Let β_m be the value of β for a predetermined $F(\beta)$, that is, q, calculated with the m-level approximation to σ. The rate of convergence of β_m to β_∞ can be estimated as follows: We want to find $\Delta\beta_m$, the change induced in β as we go back to level $m-1$. In the transfer matrix, to move back to level $m-1$ means setting all second entries in every row equal to the preceding entry. We call that matrix $T'_m(\beta)$, $T_{m-1}(\beta)$ exxpressed in the form of $T_m(\beta)$.

Table I. The Hausdorff Dimension from the Feigenbaum Construction Levels 2–9

0.535153405353073130
0.537867957190712837
0.537929308470491418
0.538033608239975927
0.538040372358731812
0.538044516455052273
0.538044941165327028
0.538045111940850554

Let ξ_R and ξ_L be the eigenvectors to the right and to the left corresponding to the largest eigenvalue of $T_m(\beta_m)$. Then, to first order

$$\delta\lambda(\beta) = \langle \xi_L | T'_m(\beta_m + \Delta\beta_m) - T_m(\beta_m) | \xi_R \rangle \tag{7.1}$$

We want to find $\Delta\beta_m$ so that $\delta\lambda(\beta) = 0$, which to first order implies

$$\frac{\Delta\beta_m}{\beta_m} \approx \frac{\langle \xi_L | \, T'_m(1) - T_m(1) \, | \xi_R \rangle}{\langle \xi_L | \, \ln T_m(1) \, | \xi_R \rangle} \tag{7.2}$$

Hence, asymptotically $\Delta\beta$ is proportional to a weighted average of the changes in σ. Therefore the sequence β_m should converge to the asymptotic value β as a sequence with overlayed geometric transients. This is also what one finds numerically. As the sequence converges geometrically, one may use the method of Shanks' transformations[12] to improve the estimate of the asymptotic value.

Successive approximations to the Hausdorff dimension D_0 are given for σ in Table I and for σ^{VSK} in Table II. Convergence is alternating and faster for σ^{VSK} than for σ, but both converge geometrically. Using the

Table II. The Hausdorff Dimension from the Construction of Vul *et al.*, Levels 2–9

0.537843517840060446
0.538103284698562080
0.538037608949288779
0.538046713784606500
0.538044902913195324
0.538045188263181811
0.538045136242254450
0.538045144882801222

method of Shanks' transformations, we have the following estimates for the asymptotic values:

$$D_0 = 0.5380451435(1) \qquad \text{from } \sigma^{\text{VSK}} \tag{7.3}$$

$$D_0 = 0.53804514(1) \qquad \text{from } \sigma \tag{7.4}$$

The result (7.3) agrees with the calculations of Bensimon *et al.*[7] and matches the first ten digits in the latest results by Grassberger.[18]

As one calculates higher order approximations to σ, cancallation errors enter as the lengths of the intervals asymptotically get exponentially small. Therefore there is an upper limit to the number of levels one might calculate from a given expansion of $g(x)$ or $g_p(x)$. For the expansion of $g(x)$ given by Lanford, which is accurate to one part in 10^{-30}, one is limited to roughly the 15th level. The calculations in this paper were all carried out, approximating $\sigma(t)$ at level m on the interval $[j/2^{m+1}, (j+1)/2^{m+1}]$ as $\sigma(j/2^{m+1}+)$, to level 9.

One may show (a derivation is given in Appendix B) that the quotients $|d_i^{(n)}/\tilde{d}_i^{(n)}|$ between intervals given in the two constructions introduced in Sections 2 and 3 stay bounded as the level index n tends to infinity. Hence the largest eigenvalue of the two transfer matrices must be the same and all generalized fractal dimensions agree.

9. CONCLUSION

Three points have been raised in this paper. The first is that fractal dimensions and scaling indices for the Feigenbaum attractor are straightforward to calculate if the trajectory scaling function is known. It is sometimes asserted[7] that fractal dimensions deal with the global structure of the attractor as opposed to the local character of the scaling function. However, such a remark misses two important points. First, the fractal dimensions are static quantities attached to an ordering of the atractor in space (a boxing algorithm essentially), while the scaling function is a dynamic quantity ordered in time. In the limit of a 2^∞-long cycle, every finite portion of $\sigma(t)$ coresponds to infinitely many iteration points scattered all over the attractor, and due to the continuity properties of σ, essentially every value of σ is attained everywhere on the attractor. σ is thus as nonlocal in space as can be. Second, the fractal dimensions follow from an averaging that disregards most of the information contained in σ. If such an averaging is not performed, the local scaling indices,[6] or pointwise fractal dimensions,[15] as function of position are much worse functions than σ, since they are discontinuous on a dense set with all discontinuities of order unity. The second point raised is that the two usual constructions given in the literature are not equivalent. The third point is that the Hausdorff dimensions calculated from the two constructions agree to seven orders of magnitude better than the scaling functions from which they have been calculated, and that all fractal dimensions of the two constructions do in fact agree.

APPENDIX A

The function $\sigma(t)$ has discontinuities on all dyadic rationals, that is, on all points $j/2^{m+1}$. The value of σ immediately after the discontinuity is

$$\begin{aligned}\sigma(j/2^{m+1}+) &\underset{l\to\infty}{=} \sigma(j\cdot 2^l+1)/2^{m+l+1} \\ &= \frac{A(i_0,..., i_m, 0, 0,..., 1) - A(i'_0,..., i_m, 0, 0,..., 1)}{A(i_1,..., i_m, 0, 0,..., 1) - A(i'_1,..., i_m, 0, 0,..., 1)} \\ &\approx \frac{g''_{m+l+1}(0)}{g''_{m+l}(0)} \frac{(A(i_0,..., i_m))^2 - (A(i'_0,..., i_m))^2}{(A(i_1,..., i_m))^2 - (A(i'_1,..., i_m))^2} \qquad \text{(A1)}\end{aligned}$$

where the prefactor goes to one as $l \to \infty$. Hence

$$\begin{aligned}&\sigma(j/2^{m+1}+) - \sigma(j/2^{m+1}) \\ &= \sigma(j/2^{m+1}) \left[\frac{A(i_0,..., i_m) + A(i'_0,..., i_m)}{A(i_1,..., i_m) + A(i'_1,..., i_m)} - 1\right] \\ &= \frac{\sigma(j/2^{m+1})}{A(i_1,..., i_m) + A(i'_1,..., i_m)} \{[A(i_0,..., i_m) - A(i_1,..., i_m)] \\ &\quad + [A(i'_0,..., i_m) + A(i'_1,..., i_m)]\} \qquad \text{(A2)}\end{aligned}$$

If the string $(i_0, i_1,..., i_m)$ ends with a tail of zeros, then these last bits will not contribute to the value of the discontinuity, as the α's to the end will cancel between the numerator and the denominator. Therefore we assume $i_m = 1$ and consider how the differences in the numerator change as we read bits from i_n up to i_m, i_n being a bit in the string below i_m. For simplicity we assume that $i_n = 1$, and only consider the first difference in the numerator.

We therefore start with the difference $A(i_0,..., i_n) - A(i_1,..., i_n)$. It is certainty less than

$$g_{n+1}(0) - g_n((1/\alpha)\, g_{n-1}(0)) \approx g(0) - g((1/\alpha)\, g(0)) + O(\delta^{-n}) \qquad \text{(A3)}$$

With the normalization $g_1(0) = 1$, the expansion of g is given by

$$g(x) \approx 1.365018 - 1.119132x^2 + 4.121082 \times 10^{-2}x^4 + \cdots$$

Approximately, then,

$$g(0) - g((1/\alpha)\, g(0)) = 0.3290... \tag{A4}$$

Hence $A(i_0,..., i_n) - A(i_1,..., i_n)$ can be considered small. Then

$$\begin{aligned}
&A(i_0,..., i_n, i_{n+1}) - A(i_1,..., i_n, i_{n+1}) \\
&\quad \approx g\{(1/\alpha)\, A(i_1,..., i_n) + (1/\alpha)[A(i_0,..., i_n) - A(i_1,..., i_n)]\} \\
&\qquad - g((1/\alpha)\, A(i_1,..., i_n)) + O(\delta^{-n-1}) \\
&\quad \approx g'((1/\alpha)\, A(i_1,..., i_n))\{(1/\alpha)[A(i_0,..., i_n) - A(i_1,..., i_n)]\} \\
&\qquad + (1/2)\, g''((1/\alpha)\, A(i_1,..., i_n)) \\
&\qquad \times \{(1/\alpha)[A(i_0,..., i_n) - A(i_1,..., i_n)]\}^2 + \cdots + O(\delta^{-n}) \qquad \text{if } i_{n+1} = 1 \\
&\quad = (1/\alpha)[A(i_0,..., i_n) - A(i_1,..., i_n)] \qquad \text{if } i_{n+1} = 0
\end{aligned} \tag{A5}$$

If $i_{n+1} = 0$, $A(i_0,..., i_n, i_{n+1}) - A(i_1,..., i_n, i_{n+1})$ is additionally small by a factor α^{-1}. If $i_{n+1} = 1$, this is almost true as

$$g'((1/\alpha)\; g((1/\alpha)\; g(0)) \leqslant g|((1/\alpha)\, A(i_1,..., i_n)) \leqslant g'((1/\alpha)\; g(0)) \tag{A6}$$

or

$$0.927... \leqslant g'((1/\alpha)\, A(i_1,..., i_n)) \leqslant 1.23... \tag{A7}$$

Hence the term linear in $\{(1/\alpha)[A(i_0,..., i_n) - A(i_1,..., i_n)]\}$ is multiplied by a term close to 1, as is the quadratic term,

$$|(1/\alpha)[A(i_0,..., i_n) - A(i_1,..., i_n)] \leqslant 0.15...$$

and therefore the quadratic and higher terms may be neglected.

Now consider

$$A(i_0,..., i_n, i_{n+1}, i_{n+2}) - A(i_1,..., i_n, i_{n+1}, i_{n+2})$$

If i_{n+1} is 1, the analysis goes through as in the preceding step, the only difference being that the approximations are better this time. We would then have that

$$\begin{aligned}
&A(i_0,..., i_n, i_{n+1}, i_{n+2}) - A(i_n, i_{n+1}, i_{n+2}) \\
&\quad \approx (\alpha^{-2}[A(i_0,..., i_n) - A(i_1,..., i_n)]
\end{aligned} \tag{A8}$$

The difference is small by another factor α^{-1}.

If, on the other hand, i_{n+1} is zero,

$$
\begin{aligned}
&A(i_0,..., i_n, i_{n+1}, i_{n+2}) - A(i_1,..., i_n, i_{n+1}, i_{n+2}) \\
&\quad \approx g'(\alpha^{-2}A(i_1,..., i_n))\{\alpha^{-2}[A(i_0,..., i_n) - A(i_1,..., i_n)]\} \\
&\qquad + (1/2)\, g''(\alpha^{-2}A(i_1,..., i_n)) \\
&\qquad \times \{\alpha^{-2}[A(i_0,..., i_n) - A(i_1,..., i_n)]\}^2 + \cdots + O(\delta^{-n}) \qquad \text{if} \quad i_{n+2} = 1 \\
&\quad = \alpha^{-2}[A(i_0,..., i_n) - A(i_1,..., i_n)] \qquad \text{if} \quad i_{n+2} = 0
\end{aligned}
\tag{A9}
$$

It is clear that the quadratic term can never become dominant, so we discard it here also.

It should now be clear how to proceed up to i_m: for every step the difference is smaller by a factor α^{-1}; if $i_k = 0$, we get an additional factor α^{-1}. Hence we deduce A.

We now consider the change in σ as we introduce the next level approximation, that is, differences of the kind

$$
\begin{aligned}
&\sigma(i_0,..., i_m, 1) - \sigma(i_0,..., i_m, 0) \\
&\quad = \frac{A(i_0,..., i_m, 1) - A(i'_0,..., i_m, 1)}{A(i_1,..., i_m, 1) - A(i'_1,..., i_m, 1)} \\
&\qquad - \frac{A(i_0,..., i_m, 0) - A(i'_0,..., i_m, 0)}{A(i_1,..., i_m, 0) - A(i'_1,..., i_m, 0)}
\end{aligned}
\tag{A10}
$$

From above we know that $A(i'_0,..., i_m) - A(i_0,..., i_m)$ and $A(i'_1,..., i_m) - A(i_1,..., i_m)$ are small at least by a factor α^{-m}. Hence

$$
\begin{aligned}
&\sigma(i_0,..., i_m, 1) - \sigma(i_0,..., i_m, 0) \\
&\approx \frac{\begin{array}{l} \big(g'(\alpha^{-1}A(i'_0,..., i_m))\{\alpha^{-1}[A(i_0,..., i_m) - A(i'_0,..., i_m)]\} \\ \quad + (1/2)\, g''(\alpha^{-1}A(i'_0,..., i_m))\{\alpha^{-1}[A(i_0,..., i_m) - A(i'_0,..., i_m)]\}^2 \\ \quad + \cdots + O(\delta^{-m})\big) \end{array}}{\begin{array}{l} \big(g'(\alpha^{-1}A(i'_1,..., i_m))\{\alpha^{-1}[A(i_1,..., i_m) - A(i'_1,..., i_m)]\} \\ \quad + (1/2)\, g''(\alpha^{-1}A(i'_1,..., i_m))\{\alpha^{-1}[A(i_1,..., i_m) - A(i'_1,..., i_m)]\}^2 \\ \quad + \cdots + O(\delta^{-m})\big) \end{array}} \\
&\qquad - \frac{A(i_0,..., i_m) - A(i'_0,..., i_m)}{A(i_1,..., i_m) - A(i'_1,..., i_m)}
\end{aligned}
\tag{A11}
$$

If we keep in mind that the sequence $(i_0,..., i_m)$ should not have a long tail of zeros, we have

$$\begin{aligned}
&\sigma(i_0,..., i_m, 1) - \sigma(i_0,..., i_m, 0)\\
&\quad\approx \sigma(i_0,..., i_m, 0)\,(-1 + O(\delta^{-m}))\\
&\qquad\times\frac{\left(1 + \dfrac{(1/2)\,g''(\alpha^{-1}A(i'_0,..., i_m))}{g'(\alpha^{-1}A(i'_0,..., i_m))}\{\alpha^{-1}[A(i_0,..., i_m) - A(i'_0,..., i_m)]\}\right)}{\left(1 + \dfrac{(1/2)\,g''(\alpha^{-1}A(i'_1,..., i_m))}{g'(\alpha^{-1}A(i'_1,..., i_m)}\{\alpha^{-1}[A(i_1,..., i_m) - A(i'_1,..., i_m)]\}\right)}\\
&\quad\approx \sigma(i_0,..., i_m, 0)\,\frac{1}{2\alpha}\,\frac{g''(\alpha^{-1}A(i_1,..., i_m))}{g'(\alpha^{-1}A(i'_1,..., i_m))}\\
&\qquad\times[\sigma(i_0,..., i_m, 0) - 1][A(i_1,..., i_m) - A(i'_1,..., i_m)]\\
&\qquad+ O(\delta^{-m})
\end{aligned}\tag{A12}$$

If the sequence $(i_0,..., i_m)$ has a tail of zeros, it does not contribute as in the former case.

Hence $\sigma(i_0,..., i_m, 1) - \sigma(i_0,..., i_m, 0)$ behaves asymptotically as $\sigma(i_0,..., i_m, +) - \sigma(i_0,..., i_m)$ and we may deduce B.

We now consider differences of the type $\sigma(i_0,..., i_m, +) - \sigma(i_0,..., i_m, 1)$. If $i_m = 1$, we have from above that this quantity is small as $\alpha^{-m}\alpha^{n(i)}$, where $n(i)$ is the number of zeros in the sequence $i = (i_0,..., i_m)$, so we assume that the sequence has a tail with zeros, and can be written $(i_0,..., i_k, 0,..., 0, i_m)$; $i_k = 1$; $j_{k+1} = \cdots = i_m = 0$. Then

$$\begin{aligned}
&\sigma(i_0,..., i_k, 0,..., i_m, +) - \sigma(i_0,..., i_m, 0,..., i_m, 1)\\
&\quad= \lim_{l\to\infty} \sigma(i_0,..., i_k, 0,..., i_m, 0, 0,..., i_l = 1)\\
&\qquad - \sigma(i_0,..., i_k, 0,..., i_m, 0, 0,..., i_l = 0)\\
&\quad= \frac{g_{l+1}(\alpha^{-l+k}A(i_0,..., i_k)) - g_{l+1}(\alpha^{-l+k}A(i'_0,..., i_k))}{g_l(\alpha^{-l+k}A(i_1,..., i_k)) - g_l(\alpha^{-l+k}A(i'_1,..., i_k))}\\
&\qquad - \frac{g_{m+2}(\alpha^{-m-1+k}A(i_0,..., i_k)) - g_{m+2}(\alpha^{-m-1+k}A(i'_0,..., i_k))}{g_{m+1}(\alpha^{-m-1+k}A(i_1,...,i_k)) - g_{m+1}(\alpha^{-m-1+k}A(i'_1,..., i_k))}
\end{aligned}\tag{A13}$$

We expand all arguments around $x=0$ and find that the difference is

$$\begin{aligned}
&\sim\sigma(i_0,..., i_m, +)\{1-[g''_{m+2}(0)/g''_{m+1}(0)]\\
&\quad\times[1+(1/12)[g''''_{m+2}(0)/g''_{m+2}(0)]((\alpha^{-m-1+k})^2\{[A(i_0,..., i_k)]^2\\
&\quad+[A(i'_0,..., i_k)]^2\})-(1/12)[g''''_{m+1}(0)/g''_{m+1}(0)]\\
&\quad\times((\alpha^{-m-1+k})^2\{[A(i_1,..., i_k)]^2+[A(i'_1,..., i_k)]^2\})]\}\\
&\approx\sigma(i_0,..., i_k, +)[O(\delta^{-m})+(1/12)[g''''_{m+1}(0)/g''_{m+1}(0)]\\
&\quad\times((\alpha^{-m-1+k})^2\{[A(i_0,..., i_k)]^2\\
&\quad-[A(i_1,..., i_k)]^2+[A(i'_0,..., i_k)]^2-[A(i'_1,..., i_k)]^2\})]
\end{aligned}\tag{A14}$$

and so the tail of zeros contributes as any other subsequence of zeros. We consider differences of the kind $\sigma(i_0,..., i_m, 1, +)-\sigma(i_0,..., i_m, 0, +)$. However, from above we know that

$$\sigma(i_0,..., i_m, 1)\approx\sigma(i_0,..., i_m, +)=\sigma(i_0,..., i_m, 0, +)\tag{A15}$$

and

$$\sigma(i_0,..., i_m, 1, +)\approx\sigma(i_0,..., i_m, 1)\tag{A16}$$

where $\approx$ means that the differences are small as $\alpha^{-m}(\alpha^{-1})^{n(i)}+O(\delta^{-m})$ regardless of the sequence. Hence the first part of D.

Similarly, if one considers differences of the kind $\sigma(2j/2^{m+2})-\sigma((2j-1)/2^{m+2})$, one knows that

$$\sigma((2j-1)/2^{m+2})\approx\sigma((2j-2)/2^{m+2})\approx\sigma(2j/2^{m+2})\tag{A17}$$

Hence C and the second part of D.

APPENDIX B

Consider

$$\frac{A(i_1,..., i_N)-A(i'_1,..., i_N)}{B(i_1,..., i_N)-B(i'_1,..., i_N)}\tag{B1}$$

If the sequence $(i_1,..., i_N)$ ends with a series of zeros, $(i_1,..., i_N)=(i_1,..., i_k=1, 0,..., 0)$, then

$$\frac{A(i_1,..., i_N)-A(i'_1,..., i_N)}{B(i_1,..., i_N)-B(i'_1,..., i_N)}=\frac{A(i_1,..., i_k)-A(i'_1,..., i_k)}{B(i_1,..., i_k)-B(i'_1,..., i_k)}\tag{B2}$$

Therefore we assume $i_N = 1$. Then

$$\frac{A(i_1,..., i_N = 1) - A(i'_1,..., i_N = 1)}{B(i_1,..., i_N = 1) - B(i'_1,..., i_N = 1)}$$

$$= \frac{\begin{aligned}&\{-(g_{N+1} \circ 1/\alpha)' A(i_1,..., i_{N-1}) \cdot [A(i'_1,..., i_{N-1}) - A(i_1,..., i_{N-1})] \\ &- (1/2)(g_{N+1} \circ 1/\alpha)'' A(i_1,..., i_{N-1})[A(i'_1,..., i_{N-1}) \\ &- A(i_1,..., i_{N-1})]^2 + \cdots\}\end{aligned}}{\begin{aligned}&\{-(g \circ 1/\alpha)' B(i_1,..., i_{N-1})[B(i'_1,..., i_{N-1}) - B(i_1,..., i_{N-1})] \\ &- (1/2)(g \circ 1/\alpha)'' B(i_1,..., i_{N-1})[B(i'_1,..., i_{N-1}) \\ &- B(i_1,..., i_{N-1})]^2 + \cdots\}\end{aligned}} \tag{B3}$$

If $(i_1,..., i_{N-1})$ does not have a long tail of zeros, this is

$$\begin{aligned}&\sim \frac{A(i_1,..., i_{N-1}) - A(i'_1,..., i_{N-1})}{B(i_1,..., i_{N-1}) - B(i'_1,..., i_{N-1})} \\ &\quad \times \left(1 + \frac{(1/2)(g \circ 1/\alpha)'' B(i_1,..., i_N)}{(g \circ 1/\alpha)' B(i_1,..., i_N)} \{[B(i_1,..., i_N) - B(i'_1,..., i_N)] \right. \\ &\quad \left. - [A(i_1,..., i_N) - A(i'_1,..., i_N)]\} + O(\delta^{-N})\right)\end{aligned} \tag{B4}$$

Otherwise, if $(i_1,..., i_{N-1}) = (i_1,..., i_k = 1, 0,..., 0)$, one expands around $x = 0$ and finds

$$\begin{aligned}&\sim \frac{(1/2)\, g''(0)(1/\alpha^{k-1})^2\{[A(i_1,..., i_k)]^2 - [A(i'_1,..., i_k)]^2\} + O(\delta^{-N}) + \cdots}{(1/2)\, g''(0)(1/\alpha^{k-1})^2\{[B(i_1,..., i_k)]^2 - [B(i'_1,..., i_k)]^2\} + \cdots} \\ &= \frac{A(i_1,..., i_k) - A(i'_1,..., i_k)}{B(i_1,..., i_k) - B(i'_1,..., i_k)} \left\{\frac{A(i_1,..., i_k) + A(i'_1,..., i_k)}{B(i_1,..., i_k) + B(i'_1,..., i_k)} \right. \\ &\quad \left. \times [1 + O(\delta^{-N})] + \cdots \right\}\end{aligned} \tag{B5}$$

Hence one may write

$$\frac{A(i_1,..., i_N) - A(i'_1,..., i_N)}{B(i_1,..., i_N) - B(i'_1,..., i_N)} \approx \prod_{k=1}^{N} (1 + x_k) \tag{B6}$$

where $x_k = 0$ for $i_k = 0$ and $|x_k| \leqslant |1/\alpha|^k$ for $i_k = 1$. When N tends to infinity, this gives an infinite product with value of order 1.

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